A NOTE ON THE DISTRIBUTION OF THE ORDER STATISTICS OF THE RATIOS OF ORDER STATISTICS WHEN THE SAMPLE SIZE IS RANDOM

BY

VISHNU DAYAL JHA
Punjabi University, Patiala
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INTRODUCTION

The distribution of the maximum and minimum of ratios of order statistics from a continuous population for fixed sample size have been investigated and used in problems of selection and ranking by Barlow, Gupta and Panchapakesan [1]. They have also obtained the distribution-free selection rules using the percentage points of these order statistics. In this paper we are to discuss the exact distributions of the order statistics of the ratios of order statistics from any arbitrary population for random sample size. The results are illustrated by assuming the truncated distribution of sample size such as binomial and Poisson.

Let X_i $(i=0,1, \ldots, p)$ be $(p+1)$ independent identically distributed non-negative random variables each representing the j-th order statistic in a random sample of size n from a c d f $F(x)$ of a non-negative random variable. Let n be not fixed in advance but a discrete random variable independent of $F(x)$. Let $P\,(n=t)=p(t)$ be the probability mass function (pmf) of n such that

$$\sum_{t=j}^{\infty} p\,(t)=1.$$

Let $F^*(x)$ be the c d f of X_i $(i=0,\ 1,\ \ldots,\ p)$. Consider the ratios $Y_i=X_i/X_o$ $(i=1,\ 2, \ldots, p)$ so that $Y_1, Y_2, \ldots, Y_p$ form a sequence of dependent exchangeable random variables. Further let us denote by $Y_{(k)}$ the k-th smallest value of $Y_1, Y_2, \ldots, Y_p$ so that $Y_{(1)} \leqslant Y_{(2)} \leqslant \ldots \leqslant Y_{(p)}$.

1. DISTRIBUTION OF X_t

For an arbitrary c d f $F(x)$

$$F^*(x)=P\,(X_t\leqslant x)$$

$$=\sum_{t=j}^{\infty} P\,(t) \sum_{k=j}^{\infty} \binom{t}{k} F^k(x)\,(1-F(x))^{t-k} \qquad \ldots(1.1)$$

$$=\sum_{t=j}^{\infty} p(t)\; I_{F(x)}\,(j,\, t-j+1)$$

where

$$I_{F(x)}(j,\, t-j+1)=\frac{1}{B\,(j\; t-j+1)} \int_0^{F(x)} u^{j-1}(1-u)^{t-j}\, du$$

is the Incomplete beta function with parameters j and $t-j+1$ and is tabled by Pearson [2]. For continuous X, the pdf of X_t is given by

$$f^*(x)=\sum_{t=j}^{\infty} p(t)\; t\,!\,((j-1)\,!\,(t-j)\,!)^{-1}\, F^{j-1}(x)\,(1-F(x))^{t-j}\, f(x) \qquad \ldots(1.2a)$$

and for discrete X, the pmf of X_t is

$$f_1^*\;(x)=\sum_{t=j}^{\infty} p(t)\; [I_{F(x)}\,(j, t-j+1)-I_{F(x-1)}\,(j,\, t-j+1)] \qquad \ldots(1.2b)$$

2. DISTRIBUTION OF $Y_{(k)}$

Let $G_{k\,:\,p}\,(y)$-be the cdf of $Y_{(k)}$ and for continuous X be given by

$$G_{k\,:\,p}(y)=\int_0^{\infty} \sum_{s=k}^{p} \binom{p}{s} (F^*(xy))^s\,(1-F^*(xy))^{p-s} f^*(x)\; dx \qquad \ldots(2.1a)$$

$$=\int_0^{\infty} I_{F^*(xy)}\,(k,\, p-k+1)\;\; f^*(x)\; dx$$

For discrete X, we have

$$G_{k\,:\,p}(y)=\sum_{x=0}^{\infty} I_{F^*(xy)}\,(k,\, p-k+1)\; f_1^*\,(x) \qquad \ldots(2.1b)$$

We can also use the following result due to Young [4] to express $G_{k:p}(y)$ in terms of $G_{r:r}(y)$ for $r \geqslant k$

$$G_{k:p}(y) = \binom{p}{k} \sum_{r=k}^{p} (-1)^{r-k} \binom{p-k}{r-1} \frac{k}{r} G_{r:r}(y) \qquad \text{...(2.2)}$$

$$= \sum_{r=k}^{p} (-1)^{r-k} \binom{r-1}{k-1} \binom{p}{r} G_{r:r}(y)$$

3. ILLUSTRATIONS (*a*)

Let n be a truncated binomial variate with pmf

$$p(t) = \frac{\binom{N}{t} \pi^t (1-\pi)^{N-t}}{\sum_{t=j}^{N} \binom{N}{t} \pi^t (1-\pi)^{N-t}}, \quad t = j, j+1, \ldots, N$$

$$= 0 \qquad \text{otherwise}$$

$$F^*(x) = \frac{I_{\pi F(x)}(j, N-j+1)}{I_\pi (j, N-j+1)}$$

For continuous X the pdf of X_i is

$$f^*(x) = \frac{N!\,(\pi F(x))^{j-1} (1-\pi F(x))^{N-j}}{(j-1)!\,(N-j)!\, I_\pi (j, N-j+1)} \pi f(x)$$

For discrete X the pmf of X_i is

$$f_1^*(x) = \frac{I_{\pi F(x)}(j, N-j+1) - I_{\pi F(x-1}(j, N-j+1)}{I_\pi (j, N-j+1)}$$

The cdf of $Y_{(k)}$ for continuous X is given by

$$G_{k:p}(y) = \sum_{r=k}^{p} (-1)^{r-k} \binom{r-1}{k-1} \binom{p}{r} \int_0^\infty F^{j-1}(x)$$

$$(1-\pi F(x))^{N-j} \sum_{s=0}^{Nr} b(s, r, N, j)$$

$$(1-\pi F(xy))^s f(x)\, dx / [B(j, N-j+1) (I_\pi (j, N-j+1))^{r+1}]$$

where $b(s, r, N, j)$ is the coefficient of y^s in

$$\left(\sum_{t=j}^{N} (1-y)^t y^{N-t} \right)^r$$

and given by Barlow, Gupta and Panchapakesan [1]

Similarly for discrete variate X we have

$$G_{k:p}(y)=\sum_{r=k}^{p}(-1)^{r-k}\binom{r-1}{k-1}\binom{p}{r}(I_{\pi}(j,\ N-j+1))^{-r-1}$$

$$=\sum_{x=0}^{\infty}\sum_{s=0}^{Nr} b(s,\ r,\ N,\ j)\ (1-\pi F(xy))^{s}$$

$$I_{\pi F(x)}(j,\ N-j+1)-I_{\pi F(x-1)}(j,\ N-j+1)]$$

(*b*) Let n be a truncated Poisson variate with pmf

$$p(t)=\frac{\exp(-\lambda)\ \lambda^{t}/t!}{\sum_{t=j}^{\infty}\exp(-\lambda)\lambda^{t}/t!},\quad t=j,\ j+1,\ldots$$

$$=0 \qquad \text{otherwise}$$

$$F^{*}(x)=P_{\lambda F(x)}(j)/P_{\lambda}(j)$$

where

$$P_{a}(j)=\frac{1}{\Gamma(j)}\int_{0}^{a}\exp(-x)\ x^{-1}\ dx$$

$$=\sum_{t=j}^{\infty}\exp(-a)\ a^{t}/t!$$

is the Incomplete Gama function and tabled by Pearson [3].

For contiuous X, the pdf of X_i is

$$f^{*}(x)=((j-1)!\ P_{\lambda}(j))^{-1}\ \lambda F(x))^{j-1}\exp(-\lambda F(x))\ \lambda f(x)$$

For discrete X, the pmf of X_i is

$$f_{1}^{*}(x)=(P_{\lambda F(x)}(j)-P_{\lambda F(x-1)}(j))/\ P_{\lambda}(j)$$

The cdf of $Y_{(k)}$ for continuous X is

$$G_{k:p}(y)=\sum_{r=k}^{p}(-1)^{r-k}\binom{r-1}{p-1}\binom{p}{r}(P_{\lambda}(j))^{-r-1}((j-1)!)^{-1}$$

$$\int_{0}^{\infty}\exp(-\lambda(rF(xy)+F(xy)))\ \lambda^{j}\ F(x)^{j-1}$$

$$\left(\sum_{t=j}^{\infty}(\lambda F(xy))^{t}/t!\right)^{r} f(x)\ dx$$

For discrete X

$$G_{k:p}(y)=\sum_{x=0}^{\infty}\sum_{r=k}^{p}(-1)^{r-k}\,(P_\lambda\,(j))^{-r-1}\binom{r-1}{k-1}\binom{p}{r}\cdot\exp\,(-r\lambda\,F(xy))$$

$$\left(\sum_{t=j}^{\infty}(\lambda\,F(xy))^t/t!\right)^r\,[P_{\lambda F(x)}\,(j)-P_{\lambda F(x-1)}\,(j)]$$

REFERENCES

[1] Barlow, R.E., Gupta, S.S. and Panchapakesan, S. (1969). : On the Distribution of the Maximum and Minimum of Ratios of Order Statistics. Ann. Math. statist. **40**, 918-34.

[2] Pearson, K. (1934) : Tables of Incomplete B-Function. Cambridge University Press.

[3] Pearson, K. (1957) : Tables of Incomplete Function. Cambridge University Press.

[4] Young, D.H. (1967) : Recurrence Relations between the P.D.F. of Order Statistics of Dependent Variables, and some Applications. Biometrika **54**, 283-92.